JOURNAL OF COMPUTATIONAL PHYSICS **147,** 229–235 (1998)
ARTICLE NO. CP986085

NOTE

On the Determination of a Velocity Field with Prescribed Vorticity[1]

Key Words: Partial differential equations 65M99; Vorticity flows 76C05; Navier-Stokes equations 76D05.

1. INTRODUCTION

One of the difficulties which one meets if one tries to solve the unsteady Navier–Stokes equations is related to the solenoidal character of the velocity field. This constraint introduces as a related force the pressure which has to be determined, in addition to the velocity field. The time derivative of the pressure is not prescribed by the Navier–Stokes equations. The pressure is usually determined at each time step via a Poisson equation [1].

An alternative would be to eliminate the pressure by applying the curl-operator to the Navier–Stokes equation and using the resulting Helmholtz vortex equation instead. As this equation still contains the velocity, one meets the additional problem of determing the velocity components from the vorticity. For two-dimensional problems one may use the stream function and has then to solve a Poisson equation to determine the stream function from the vorticity. This leads to the often used vorticity-stream-function formulation [1, Chap. 17]. It is an important alternative to the solution in primitive variables. For three-dimensional problems one has, instead, of a stream function a vector potential with three components. One then has to solve three Poisson equations (i.e. one vector Poisson equation) if one wants to determine the vector potential from the vorticity. An alternative, which is also used [2], in which a relation between the Laplacian of the velocity and the curl of the vorticity is used requires again the solution of a vector Poisson equation. Another alternative, which has been proposed in [3] is to solve numerically the system of coupled equations, which relate the velocity to the vorticity, the so-called Cauchy–Riemann equations [4]; see also [5]. A recent review of these methods can be found in [6]. Because of these complications, the use of the primitive variables is in most cases preferred for three-dimensional problems.

[1] This is a slightly condensed form of a paper presented at the *Third European Fluid Mechanics Conference, Göttingen, September 15–18, 1997*.

0021-9991/98 $25.00

The purpose of this note is to show that the determination of one velocity field with prescribed vorticity is an easy task (i.e. linear in the number of unknowns) if a method that is well known in the theory of differential forms, namely the Flanders integral [7], is used. It is shown that this method which requires the evaluation of line integrals over homotopies is for a rectangular grid very efficient if a suitable homotopy is used. The velocity field which fulfills the apropriate boundary conditions is then obtained by adding a suitable potential flow, which requires the solution of *one* Poisson equation, i.e. the same computational effort as in primitive variables.

In retrospect, the result of the scheme is obvious. It is clear, that the velocity field is determined by the vortices only up to a gradient of some scalar. One may choose this scalar in such a way that one velocity component vanishes. The determination of the other velocity components from the the definition of the vorticity is then trivial. This leads to a velocity field with one vanishing component which has the prescribed vortices. This simple interpretation of the homotopy integral is applicable for the homotopy considered here and for similar homotopies in other coordinate systems. In general such an interpretation will not be possible.

The method is described here for three-dimensional problems. It can be applied also to two-dimensional situations. There, there are no obvious advantages compared to the vorticity-stream-function formulation. A reader, who is not interested in the mathematical background and possible generalizations and who wants to avoid the use of differential forms may omit the paragraphs between (5) and (6) and continue after (5) immediately with (6).

In the applications one usually has to determine a velocity field with known vorticity and known normal component of the velocity. This problem is called P(i) in [3] and is denoted as (1) in [5]. It has to be solved in each time step for an unsteady computation and in each iteration for a steady computation. In [5] Wu, Wu, and Wu write, that solving this problem is usually the most time-consuming part of the whole computation. They recommend solving only two components of the vector Poisson equation and determining the third velocity component from the continuity equation. Then only two Poisson equations, instead of the ordinary three, have to be solved. This leads to a 33% reduction in CPU time. As we have to solve only one Poisson equation, an additional saving of 50% should be possible.

2. BASIC RELATIONS

Let $\mathbf{u}(\mathbf{x})$ be a three-dimensional solenoidal vector field which we call velocity, i.e.

$$\mathbf{u}(\mathbf{x}) = (u_1, u_2, u_3) \quad \text{with } \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{1}$$

and let $\mathbf{w}$ be its vorticity field,

$$\mathbf{w}(\mathbf{x}) = \operatorname{curl} \mathbf{u}, \quad \mathbf{w} = (w_1, w_2, w_3) = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right). \tag{2}$$

Then it is easy to determine $\mathbf{w}$ from $\mathbf{u}$. To solve the inverse problem, i.e. to determine from a solenoidal field $\mathbf{w}$ a solenoidal vector field $\mathbf{u}$, is more complicated. Usually one uses the identity

$$\operatorname{curl} \operatorname{curl} \mathbf{b} = \operatorname{grad} \operatorname{div} \mathbf{b} - \Delta \mathbf{b}, \tag{3}$$

which is valid for an arbitrary vector field, to derive from (2) a vector Poisson equation. Two methods are in use. One can introduce a vector potential **b** which fulfills the equations

$$\mathbf{u} = \operatorname{curl} \mathbf{b}, \quad \operatorname{div} \mathbf{b} = 0$$

and one obtains from (3)

$$\Delta \mathbf{b} = -\mathbf{w}, \tag{4}$$

or one applies the curl-operator to (2) and one obtains

$$\Delta \mathbf{u} = -\operatorname{curl} \mathbf{w}. \tag{5}$$

Equations (4), (5) represent both a system of three uncoupled Poisson equations for **b**, or **u**, respectively. The vectors have additionaly to fulfill the side condition of being solenoidal. In free space (4) and (5) can be solved with the Poisson integral which leads to the Biot–Savart law and for more complicated situations effective numerical schemes for their solution are often available.

Here we want to describe a method used in the theory of differential forms (see [7]) which gives directly a solution of (2) without a transformation to a vector Poisson equation. This solution is, in general, not solenoidal and does not fulfill prescribed boundary conditions. Therefore it is necessary to add to this solution an irrotational flow, which can be obtained from solving *one* Poisson equation. A reader who is not interested in the mathematical background may skip the next sections and continue directly with (6).

One introduces a differential form

$$\omega = u_1\, dx_1 + u_2\, dx_2 + u_3\, dx_3.$$

Then one obtains another form ω_w associated with the vorticity

$$\omega_w = d\omega = w_1\, dx_2\, dx_3 + w_2\, dx_3\, dx_1 + w_3\, dx_1\, dx_2.$$

One has $d\omega_w = 0$ and one looks for a form $\omega_w = d\alpha$. It is shown in [7] that α can be obtained by an integral, if there is a homotopy which consists of three functions $X_i(\tau, x_j)$ with $X_i(0, x_j) = 0$, $X_i(1, x_j) = x_j$, and ω_w is, after the replacement $x_i \to X_i(\tau, x_j)$, regular for all τ with $0 \le \tau \le 1$. Then one may use this replacement to transform ω_w to the new variables x_j, where differentials of τ are also included, and omit all contributions which do not contain $d\tau$. The obtained form $\tilde{\omega}_w$ will then be of the form

$$\tilde{\omega}_w = \tilde{w}_1\, d\tau\, dx_1 + \tilde{w}_2\, d\tau\, dx_2 + \tilde{w}_3\, d\tau\, dx_3$$

and it is shown in [7] that the form α defined by

$$\alpha = \int_0^1 \tilde{w}_1\, d\tau\, dx_1 + \int_0^1 \tilde{w}_2\, d\tau\, dx_2 + \int_0^1 \tilde{w}_3\, d\tau\, dx_3$$

is related to ω_w by $\omega_w = d\alpha$. This means $d(\omega - \alpha) = 0$. One has, therefore,

$$\omega - \alpha = d\chi$$

with some function χ; the vector field defined by α differs from **u** only by a gradient.

The form α depends, in general, on the homotopy $X_i(\tau, x_j)$. We consider transformations

$$H_{x_1}(\tau): (x_1 \to \phi(\tau)\, x_1, x_2 \to x_2, x_3 \to x_3)$$

and, similarly, H_{x_2} and H_{x_3} which are adapted to a rectangular coordinate system. The function $\phi(\tau)$ is a monotonically increasing smooth function which vanishes for $\tau < 0$ and is equal to 1 for $\tau > 1$. We can then introduce a homotopy,

$$H(\tau) = H_{x_3}(3\tau - 2) H_{x_2}(3\tau - 1) H_{x_1}(3\tau).$$

This leads to

$$\tilde{\omega}_w = \begin{cases} 0, \\ w_3(x_1, \phi(3\tau - 1)x_2, 0)3x_2\phi'(3\tau - 1)\, dx_1 d\tau, \\ (w_1(x_1, x_2, \phi(3\tau - 2)x_3)\, dx_2 - w_2(x_1, x_2, \phi(3\tau - 2)x_3)\, dx_1)3x_3\phi'(3\tau - 2)\, d\tau, \end{cases}$$

in $\tau < \frac{1}{3}$, $\frac{1}{3} < \tau < \frac{2}{3}$, and $\frac{2}{3} < \tau < 1$, respectively. One then obtains for α:

$$\alpha = -\int_0^{x_2} w_3(x_1, \xi, 0)\, d\xi\, dx_1 - \int_0^{x_3} w_1(x_1, x_2, \xi)\, d\xi\, dx_2 + \int_0^{x_3} w_2(x_1, x_2, \xi)\, d\xi\, dx_1.$$

Therefore, the velocity field

$$\tilde{\mathbf{u}} = \left(\int_0^{x_3} w_2(x_1, x_2, \xi)\, d\xi - \int_0^{x_2} w_3(x_1, \xi, 0)\, d\xi, -\int_0^{x_3} w_1(x_1, x_2, \xi)\, d\xi, 0 \right), \quad (6)$$

which has a vanishing third component, has the vorticity (w_1, w_2, w_3). This is trivial to verify for the first two components. For the third component one obtains

$$(\text{curl}\, \tilde{\mathbf{u}})_3 = -\int_0^{x_3} \left(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} \right)(x_1, x_2, \xi)\, d\xi + w_3(x_1, x_2, 0),$$

which agrees with w_3 if $\mathbf{w}$ is solenoidal. This is obvious, if one replaces the integrand by $-\partial w_3/\partial x_3$. In general, one obtains from (6)

$$\text{curl}\, \tilde{\mathbf{u}} = (w_1, w_2, w_3) - \left(0, 0, \int_0^{x_3} \Theta(x_1, x_2, \xi)\, d\xi \right) \quad \text{with } \Theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}$$

which agrees with $\mathbf{w}$ if Θ vanishes, as it should. For a nonsolenoidal $\mathbf{w}$ (6) solves the equation

$$\text{curl}\, \tilde{\mathbf{u}} = \mathbf{w} + \theta \mathbf{e}_3 \quad \text{or} \quad (\text{curl}\, \tilde{\mathbf{u}} - \mathbf{w}) \times \mathbf{e}_3 = 0$$

with some arbitrary function θ and where $\mathbf{e}_3$ is the unit vector in x_3-direction and the second relation denotes a vector product. One may state that the equation which is solved if $\mathbf{w}$ is not solenoidal is the projection of (2) into the x_1-x_2 plane. Of course, the direction $\mathbf{e}_3$ is completely arbitrary. In situations, where the velocity component in one direction is small compared to the velocity component in another direction, it is certainly useful to choose this direction as the direction of $\mathbf{e}_3$.

To extend the method to other coordinate systems, one might mention that (2) is valid, not only in rectangular coordinates. It is shown in tensor calculus books, that (2) is valid in all coordinate systems, if natural basis vectors are used (compare e.g., [8, Eq. (1.4.10)). The more complicated expressions, which one usually sees, originate from the use of basis vectors of unit length. It is not difficult to rewrite, e.g., the expressions for the velocity and vorticity components in polar coordinates ϑ, ϕ, r as

$$(r \sin\vartheta\, w_\vartheta, r w_\phi, r^2 \sin\vartheta\, w_r) = \left(\frac{\partial u_r}{\partial \phi} - \frac{\partial r \sin\vartheta\, u_\phi}{\partial r}, \frac{\partial r u_\vartheta}{\partial r} - \frac{\partial u_r}{\partial \vartheta}, \frac{\partial r \sin\vartheta\, u_\phi}{\partial \vartheta} - \frac{\partial r u_\vartheta}{\partial \phi}\right).$$

These equations are obviously of the form of (2) if one writes $w_1 = r \sin\vartheta\, w_\vartheta$ etc. and (6) can be used also in polar coordinates to determine the vorticity from the velocity.

Having obtained one solution of (2), it is easy to obtain the solution $\mathbf{u}$ of (2). It differs from $\tilde{\mathbf{u}}$ only by an irrotational velocity field; i.e.,

$$\mathbf{u} = \tilde{\mathbf{u}} + \text{grad}\, \chi \tag{7}$$

with a certain smooth function χ. The function χ can be determined if the divergence of $\mathbf{u}$ is known,

$$\text{div}\, \mathbf{u} = \rho, \tag{8}$$

in a bounded domain D and if apropriate boundary conditions are prescribed, e.g.

$$\mathbf{u} \cdot \mathbf{n} = u_\Gamma \tag{9}$$

on the boundary Γ of D. Then χ can be determined from a Poisson equation

$$\Delta\chi = \rho - \text{div}\, \tilde{\mathbf{u}}, \quad \mathbf{n} \cdot \nabla\chi = u_\Gamma - \tilde{\mathbf{u}} \cdot \mathbf{n}. \tag{10}$$

Integrating (8) over the domain D leads with Gauss' theorem to

$$\int_D \text{div}\, \mathbf{u}\, d^3x = \oint_\Gamma u_\Gamma\, d^2x = \int_D \rho\, d^3x. \tag{11}$$

This relation between the functions ρ and u_Γ has to be fulfilled for a solution of (10) to exist. Conversely the function χ is not uniquely determined. An arbitrary constant can be added to any solution χ.

3. NUMERICAL APPLICATION

As a check for the numerical feasibility of the results we take the first test problem of [3] and determine the velocity of Howarth's stagnation point flow [9],

$$u_1 = x_1 f'(\eta), \quad u_2 = -R^{-1/2}(f(\eta) + \beta g(\eta)), \quad u_3 = x_3 g'(\eta), \tag{12}$$

in the box $0 \le x_1, x_3 \le 1$ and $0 \le \eta \le 5$ from its vorticity and its normal components. R

TABLE I
Results of the Test Problem Obtained Here and in Ref. [3]

Number of cells	$\frac{\|E(\mathbf{u})\|}{\|\mathbf{u}\|}$	Number of cells [3]	$\frac{\|E(\mathbf{u})\|}{\|\mathbf{u}\|}$ [3]
8^3	1.4×10^{-3}	10^3	7.2×10^{-3}
16^3	4.1×10^{-4}	20^3	1.7×10^{-3}
32^3	1.0×10^{-4}	30^3	7.3×10^{-4}
64^3	2.6×10^{-5}		

denotes the Reynolds number, $\mathrm{R} = 100$, and $\eta = \mathrm{R}^{1/2} x_2$. The parameter β is, as in [3], $\beta = \frac{1}{2}$. We discretize the equations on the rectangular staggered grid from [5] and replace all derivatives by central differences. The error will then be $O(h^2)$ if h is the grid size. The integrals in (6) are converted to sums and the Poisson equation (10) is converted to a system of linear equations, which we have solved by FFT methods. For simplicity we take for f and g

$$f(\eta) = \eta + a(1 - e^{-\eta/a}), \quad g(\eta) = \eta + b(1 - e^{-\eta/b}) \quad \text{with } a = 0.65, b = 0.75 \tag{13}$$

which differ in the computational domain by less than 5% from the exact f and g. The error is then the difference between the numerical value and the value obtained from (12). In Table I we list the values of the L_2 norms of the relative errors $\|E(\mathbf{u})\|/\|\mathbf{u}\|$, i.e. the root mean square of the values in the cells, for various numbers of cells and have shown for comparison the results of [3] which were obtained by a much more complicated compact $O(h^2)$-scheme. The data show, that the error decays approximately by a factor of 4 if the grid size is halved. This is true for both schemes and the accuracy is comparable.

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Received August 14, 1997; revised July 30, 1998

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